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Asymptotic Behavior and Oscillations of Second Order Integro-differential Equations with Deviating Argument*

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Abstract: The main object of this paper is to establish some new integral inequalities, which can be used for analyzing the asymptotic behavior and the oscillation of the solutions of some second order integro-differential equations with deviating argument.

Keywords: asymptotic behavior of solutions; second order integro-differential equations; integral inequalities

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1 Introduction

In this paper we consider the following integro-differential equation

$$\begin{aligned}&(ax')'+bx'+cx\\&=f\Big[t,x(t),x'(t),x\big(\alpha(t)\big),x'\big(\alpha(t)\big),\int_0^t g\big(t,s,x(s),x'(s),x(\beta(s)),x'(\beta(s))\big)\mathrm{d}s\Big].\end{aligned}\tag{1}$$

Here $a=a(t)$ is a positive and continuously differentiable function on $R_+=[0,\infty)$ such that $a(0)=1$; $b(t),c(t)$ are continuous functions on R_+; f and g are continuous on $R_+\times R^5$; $R_+^2\times R^4$, respectively; $\alpha(t)$, $\beta(t)$ are continuously differentiable satisfying that $\alpha(t)\leqslant t$, $\beta(t)\leqslant t$; $\alpha'(t)>0$, $\beta'(t)>0$ and $\alpha(t)$, $\beta(t)$ are eventually positive. We assume the local existence of solutions of the Cauchy problem of equation (1).

In 2004, Meng[1] studied the asymptotic behavior of solution to

$$(ax')'+bx'+cx=f\Big[t,x(t),x'(t),x\big(\varphi(t)\big),x'\big(\varphi(t)\big),\int_0^t g\big(t,s,x(s),x'(s)\big)\mathrm{d}s\Big].\tag{2}$$

His results extended those in [2-5].

The purpose of this paper is to establish a new integral inequality containing a deviating argument and to study the asymptotic behavior and the oscillation of the solutions of (1).

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We shall prove, under certain conditions, that the solution of (1) behaves asymptotically like a solution to the linear equation

$$(ax')' + bx' + cx = 0, \quad t \in R_+. \tag{3}$$

As a consequence, we obtain an explicit asymptotic behavior of the solutions of some special types of (1) which shows the oscillatory of those solutions.

We recall that the initial value problem of (1) is defined as follows (see [7]).

Let

$$\inf_{t\in[0,\infty)} \alpha(t) = \gamma_1, \quad \inf_{t\in[0,\infty)} \beta(t) = \gamma_2.$$

Let $[\gamma, 0] = [\gamma_1, 0] \cap [\gamma_2, 0]$ and θ be a given differential function defined on $[\gamma, 0]$. The initial value problem of (1) in the interval $[0, \tau)$ is to find a function $x(t)$, such that

(i) $x(0^+) = \theta(0^-)$, $x'(0^+) = \theta'(0^-)$;

(ii) $x^{(j)} = \theta^{(j)}$, $t \in [\gamma, 0]$, $j = 0, 1$;

(iii) $(a(t)x')'$ exist on $[0, \tau)$ and (1) is satisfied.

We also assume that the initial value problem of (1) is local existence. In our case, the technique depends on a generalization of the famous Gronwall-Bellman and Bihari integral inequalities to include the case of a deviating argument, which extends the inequalities used in [1-5].

As in [2,3], we suppose that the general solution of (3) is known and for any two linearly independent solutions Z_1, Z_2 of (3), we define

$$\xi(t) = |Z_1| + |Z_2|, \quad \eta(t) = |Z_1'| + |Z_2'|, \quad N(t) = \frac{\xi(t)}{a(t)W(t)},$$

where $W(t)$ denotes the Wronskian of Z_1 and Z_2;

$$W(t) = Z_1 Z_2' - Z_1' Z_2 \neq 0, \quad t \in R_+.$$

2 An integral inequality with deviating argument

In the sequel we will require the following integral inequality, which generalizes those integral inequalities used in [2-5].

Lemma 1 Suppose that the following conditions are satisfied.

1) $u(t)$ and $a(t) : R_+ \to R_+$ are continuous functions and $a(t) \geqslant 1$, $u(t)$ and $a(t)$ are all nondecreasing on R_+;

2) The functions $f(t,s)$, $g(t,s)$, $p(t,s)$, $q(t,s)$, $h(t,s)$ and $m(t,s) : D = \{(t,s) : 0 \leqslant s \leqslant t < \infty\} \to R_+$ are continuous and nondecreasing in t for each fixed $s \in R_+$;

3) $\alpha(t)$, $\beta(t)$ are continuous differentiable satisfying that $\alpha(t) \leqslant t$, $\beta(t) \leqslant t$; $\alpha'(t) > 0$, $\beta'(t) > 0$ and $\alpha(t)$, $\beta(t)$ are eventually positive;

4) The following inequality holds for all $t \in R_+$, where r, $p \in (0,1]$ are constants

$$u(t) \leqslant a(t) + \int_0^t f(t,s)u(s)\mathrm{d}s + \int_0^t g(t,s)\big[u(\alpha(s))\big]^r \mathrm{d}s$$
$$+ \int_0^t p(t,s)\Big(\int_0^s q(s,r)u(r)\mathrm{d}r\Big)\mathrm{d}s + \int_0^t h(t,s)\Big(\int_0^s m(s,r)\big[u(\beta(r))\big]^p \mathrm{d}r\Big)\mathrm{d}s. \quad (4)$$

Then we have

$$u(t) \leqslant a(t)E(t)F(t)G(t), \quad t \in R_+. \quad (5)$$

Here

$$E(t) = \begin{cases} \big[1 + (1-r)\int_0^{\alpha^{-1}(t)} g(t,s)\mathrm{d}s\big]^{\frac{1}{1-r}}, & 0 < r < 1; \\ \exp\big(\int_0^{\alpha^{-1}(t)} g(t,s)\mathrm{d}s\big), & r = 1, \end{cases}$$

$$F(t) = \exp\Big(E(t)\int_0^t \big(f(t,s) + P(t)q(t,s)\big)\mathrm{d}s\Big),$$

$$G(t) = \begin{cases} \big[1 + (1-p)\int_0^{\beta^{-1}(t)} E(t)F(t)H(t)m(t,s)\mathrm{d}s\big]^{\frac{1}{1-p}}, & 0 < p < 1; \\ \exp\big(\int_0^{\beta^{-1}(t)} E(t)F(t)H(t)m(t,s)\mathrm{d}s\big), & p = 1, \end{cases}$$

$$P(t) = \int_0^t p(t,s)\mathrm{d}s, \quad H(t) = \int_0^t h(t,s)\mathrm{d}s,$$

and $\alpha^{-1}(t)$, $\beta^{-1}(t)$ are the inverse of $\alpha(t)$, $\beta(t)$.

Proof By assumptions, we observe that

$$\int_0^t h(t,s)\Big[\int_0^s m(s,r)\big[u(\beta(r))\big]^p \mathrm{d}r\Big]\mathrm{d}s \leqslant H(t)\int_0^t m(t,s)\big[u(\beta(s))\big]^p \mathrm{d}s. \quad (6)$$

In fact, $u(t)$, $\beta(t)$ are all nondecreasing, $h(t,s)$ is nondecreasing in t for each fixed $s \in R_+$, so we have

$$\begin{aligned}\int_0^t m(t,s)\Big[\int_0^s h(s,r)\big[u(\beta(r))\big]^p \mathrm{d}r\Big]\mathrm{d}s &\le \int_0^t \Big[m(t,s)\big[u(\beta(s))\big]^p \int_0^s h(s,r)\mathrm{d}r\Big]\mathrm{d}s \\ &\le \int_0^t h(t,r)\mathrm{d}r \int_0^t m(t,s)\big[u(\beta(s))\big]^p \mathrm{d}s \\ &\le H(t)\int_0^t m(t,s)\big[u(\beta(s))\big]^p \mathrm{d}s.\end{aligned}$$

Using the similar method we can prove (7), so we omit it here.

$$\int_0^t p(t,s)\Big(\int_0^s q(s,r)u(r)\mathrm{d}r\Big)\mathrm{d}s \leqslant P(t)\int_0^t q(t,s)u(s)\mathrm{d}s. \quad (7)$$

Obviously, inequality (5) is valid when $t = 0$. Now we fix an arbitrary number $T \in R_+$. Then from (4) we get

$$\begin{aligned} u(t) \leqslant\ & a(T) + \int_0^t f(T,s)u(s)\mathrm{d}s + \int_0^t g(T,s)\big[u(\alpha(s))\big]^r \mathrm{d}s \\ & + P(T)\int_0^t q(T,s)u(s)\mathrm{d}s + H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s, \quad 0 \le t \le T. \end{aligned} \tag{8}$$

Let

$$\begin{aligned} A(t,T) =\ & a(T) + \int_0^t f(T,s)u(s)\mathrm{d}s + P(T)\int_0^t q(T,s)\big[u(s)\big]\mathrm{d}s \\ & + H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s, \end{aligned} \tag{9}$$

then from (8), (9) we have

$$u(t) \leqslant A(t,T) + \int_0^t g(T,s)\big[u(\alpha(s))\big]^r \mathrm{d}s, \quad 0 \leqslant t \leqslant T. \tag{10}$$

We can prove that

$$u(t) \leqslant A(t,T)B(t,T), \quad t \in [0,T], \tag{11}$$

where

$$B(t,T) = \begin{cases} \big[1 + (1-r)\int_0^{\alpha^{-1}(t)} g(T,s)\mathrm{d}s\big]^{\frac{1}{1-r}}, & 0 < r < 1; \\ \exp\big(\int_0^{\alpha^{-1}(t)} g(T,s)\mathrm{d}s\big), & r = 1. \end{cases}$$

In fact, fixing an arbitrary number $\bar{T}$ from $[0,T]$ then since $A(\bar{T},T) \geqslant 1$ and $0 < r \leqslant 1$ we obtain from (10) that

$$\frac{u(t)}{A(\bar{T},T)} \leqslant 1 + \int_0^t g(T,s)\Big[\frac{u(\alpha(s))}{A(\bar{T},T)}\Big]^r \mathrm{d}s, \quad t \in [0,\bar{T}]. \tag{12}$$

Now (12) is a special case of the well-known Bihari inequality. By simple calculation using the results in [6], we have

$$u(t) \leqslant A(\bar{T},T)B(t,T), \quad t \in [0,\bar{T}]. \tag{13}$$

Setting $t = \bar{T}$ in (13), then we obtain from (13) that

$$u(\bar{T}) \leqslant A(\bar{T},T)B(\bar{T},T). \tag{14}$$

Since $\bar{T}$ is an arbitrary number from $[0,T]$, the validity of (11) is proved. In view of $B(T,T) = E(T)$ then we obtain from (11) that

$$\begin{aligned} u(t) \leqslant A(t,T)E(T) =\ & E(T)\Big(a(T) + \int_0^t f(T,s)u(s)\mathrm{d}s \\ & + P(T)\int_0^t q(T,s)u(s)\mathrm{d}s + H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s\Big). \end{aligned} \tag{15}$$

Let

$$C(t,T) = E(T)a(T) + E(T)H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s, \quad t \in [0,T],$$

we have

$$u(t) \leqslant C(t,T) + E(T)\int_0^t \big(f(T,s) + P(T)q(T,s)\big)u(s)\mathrm{d}s.$$

By the Bellman inequality, we have

$$u(t) \leqslant C(t,T)\exp\Big(E(T)\int_0^t \big(f(T,s)+P(T)q(T,s)\big)\mathrm{d}s\Big),$$

that is

$$u(t) \leqslant \Big(E(T)a(T) + E(T)H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s\Big)F(T). \tag{16}$$

Since $\beta(t) \leqslant t$ by (16), we get that

$$u\big(\beta(t)\big) \leqslant \Big(E(T)a(T) + E(T)H(T)\int_0^t m(T,s)\big[u(\beta(s))\big]^p \mathrm{d}s\Big)F(T). \tag{17}$$

Since $a(T)E(T)F(T) \geqslant 1$ from (17) and using the same methods as above we have

$$u(t) \leqslant a(T)E(T)F(T)D(t,T), \quad t \in [0,T], \tag{18}$$

where

$$D(t,T) = \begin{cases} \big[1 + (1-p)\int_0^{\beta^{-1}(t)} E(T)F(T)H(T)m(T,s)\mathrm{d}s\big]^{\frac{1}{1-p}}, & 0 < p < 1; \\ \exp\Big(\int_0^{\beta^{-1}(t)} E(T)F(T)H(T)m(T,s)\mathrm{d}s\Big), & p = 1. \end{cases}$$

Setting $t = T$ in (18), and in view of $D(T,T) = G(T)$ we obtain

$$u(T) \leqslant a(T)E(T)F(T)G(T).$$

Since T is an arbitrary number fron R_+, the proof of this lemma is now completed.

Remark 1 The inequality (4) generalizes the inequalities used in [1-5].

3 Asymptotic behavior

Theorem 1 Assume that

1) $f \in C(R \times R^5, R)$,

$$|f(t,x_1,x_2,x_3,x_4,x_5)| \leqslant e_1(t) + r_1(t)|x_1| + r_2(t)|x_2| + r_3(t)|x_3|^p + r_4(t)|x_4|^p + r_5(t)|x_5|,$$

where $0 < p \leqslant 1$, $e_1 \in C(R_+, R_+)$, $r_i \in (R_+, R_+)$, $i = 1,2,3,4,5$;

2) $|g(t,s,x_1,x_2,x_3,x_4)| \leqslant e_2(t)+e_3(s)+k_1(t,s)|x_1|+k_2(t,s)|x_2|+k_3(t,s)|x_3|^q+k_4(t,s)|x_4|^q$, where $0 < q \leqslant 1$, $e_2, e_3 \in C(R_+, R_+)$, $k_i(t,s) \in (R_+^2, R_+)$, $i = 1,2,3,4$ are continuous functions and are nondecreasing in t for fixed $s \in R_+$;

3) The functions $N(t)e_1(t)$, $N(t)r_1(t)\xi(t)$, $N(t)r_2(t)\eta(t)$, $N(t)r_3(t)\xi^p(\alpha(t))$, $N(t)r_4(t)\eta^p(\alpha(t))$, $N(t)r_5(t)$ belong to $L_1(0,\infty)$, where $\xi(t)$, $\eta(t)$, $N(t)$ are defined in section 1;

4) The following integrals are bounded as $t \to \infty$.

$$\int_0^t N(s)r_5(s)\big[[e_1(s)+e_2(m)]\mathrm{d}m\big]\mathrm{d}s,$$

$$\int_0^t \big[k_3(t,s)[\xi^q(\beta(s))]+k_4(t,s)[\eta^q(\beta(s))]\big]\mathrm{d}s,$$

$$\int_0^t \big[k_1(t,s)\xi(s)+k_2(t,s)\eta(s)\big]\mathrm{d}s.$$

Then for any initial function $\theta(t)$ defined on $[\gamma,0]$ there is a solution $x(t)$ of (1) defined on the interval $[\gamma,0]\cup R_+$ which can be written in the form

$$x(t)=A(t)Z_1(t)+B(t)Z_2(t), \tag{19}$$

satisfying the interval condition $x(t)=\theta(t),\ t\in[\gamma,0]$, where the limits of $A(t)$ and $B(t)$ exist as $t\to\infty$.

Proof Let $x(t)$ be any solution of (1) on the interval $[\gamma,0]\cup[0,\beta)$ and satisfying the initial condition $x(t)=\theta(t),\ t\in[\gamma,0]$, here $0<\beta\leqslant+\infty$. Assume that $x(t)$ can be written in the form (19), since there are two functions to be chosen, so we may require

$$A'(t)Z_1(t)+B'(t)Z_2(t)=0. \tag{20}$$

Differentiating the both sides of (19) and by (20), we get

$$x'(t)=A(t)Z_1'(t)+B(t)Z_2'(t), \tag{21}$$

and

$$x''(t)=A'(t)Z_1'(t)+B'(t)Z_2'(t)+A(t)Z_1''(t)+B(t)Z_2''(t),$$

then

$$A'(t)Z_1'(t)+B'(t)Z_2'(t)=K(t),\quad t\in[0,\beta), \tag{22}$$

where

$$K(t)=\frac{1}{a(t)}f\Big[t,x(t),x'(t),x(\alpha(t)),x'(\alpha(t)),\int_0^t g(t,s,x(s),x'(s),x(\beta(s)),x'(\beta(s)))\mathrm{d}s\Big],$$

herein $x(t)$, $x'(t)$ are given by (19) and (21), respectively. Solving $A'(t)$ and $B'(t)$ from (20) and (22), and then integrating them from 0 to t we get

$$A(t)=A(0)+\int_0^t\frac{K(s)Z_2(s)}{W(s)}\mathrm{d}s,\quad B(t)=B(0)+\int_0^t\frac{K(s)Z_1(s)}{W(s)}\mathrm{d}s.$$

Setting $Q(t)=|A(t)|+|B(t)|,\ t\in[0,\beta)$. Then we have

$$Q(t)\leqslant Q(0)+\int_0^t\frac{K(s)(|Z_1(s)|+|Z_2(s)|)}{W(s)}\mathrm{d}s. \tag{23}$$

Applying conditions 1), 2) of Theorem 1 to (23), we have

$$\begin{aligned}
Q(t) &\leqslant Q(0)+\int_0^t \frac{\xi(s)}{W(s)a(s)}\Big|f\Big[s,x(s),x'(s),x(\alpha(s)),x'(\alpha(s)),\\
&\qquad\qquad \int_0^s g(s,m,x(m),x'(m),x(\beta(m)),x'(\beta(m)))\mathrm{d}m\Big]\Big|\mathrm{d}s\\
&\leqslant Q(0)+\int_0^t N(s)\big[r_1(s)Q(s)\xi(s)+r_2(s)Q(s)\eta(s)+r_3(s)[\xi(\alpha(s))]^p[Q(\alpha(s))]^p\\
&\qquad\qquad +r_4(s)[\eta(\alpha(s))]^p[Q(\alpha(s))]^p+e_1(s)\big]\mathrm{d}s\\
&\quad+\int_0^t N(s)r_5(s)\Big[\int_0^s [e_2(s)+e_3(m)+k_1(s,m)Q(s)\xi(m)+k_2(s,m)Q(m)\eta(m)\\
&\qquad\qquad +k_3(s,m)[\xi(\beta(m))]^q[Q(\beta(m))]^q+k_4(s,m)[\eta(\beta(m))]^q[Q(\beta(m))]^q\mathrm{d}m\Big]\mathrm{d}s\\
&\leqslant M(t)+\int_0^t N(s)\big[r_1(s)\xi(s)+r_2(s)\eta(s)\big]Q(s)\mathrm{d}s\\
&\quad+\int_0^t N(s)\big[r_3(s)[\xi(\alpha(s))]^p+r_4(s)[\eta(\alpha(s))]^p\big][Q(\alpha(s))]^p\mathrm{d}s\\
&\quad+\int_0^t N(s)r_5(s)\Big[\int_0^s (k_1(s,m)\xi(m)+k_2(s,m)\eta(m))Q(m)\mathrm{d}m\Big]\mathrm{d}s\\
&\quad+\int_0^t N(s)r_5(s)\Big[\int_0^s (k_3(s,m)[\xi(\beta(m))]^q+k_4(s,m)[\eta(\beta(m))]^q)[Q(\beta(m))]^q\mathrm{d}m\Big]\mathrm{d}s,
\end{aligned}$$

where

$$M(t)=Q(0)+\int_0^t N(s)e_1(s)\mathrm{d}s+\int_0^t N(s)r_5(s)\Big(\int_0^s [e_2(s)+e_3(m)]\mathrm{d}m\Big)\mathrm{d}s+1.$$

Then from Lemma 1 we have the following result

$$Q(t)\leqslant M(t)E(t)F(t)G(t),\quad t\in R_+, \tag{24}$$

where

$$E(t)=\begin{cases}\big[1+(1-p)\int_0^{\alpha^{-1}(t)} N(s)[r_3(s)[\xi(\alpha(s))]^p+r_4(s)[\eta(\alpha(s))]^p]\mathrm{d}s\big]^{\frac{1}{1-p}}, & 0<p<1;\\ \exp\big(\int_0^{\alpha^{-1}(t)} N(s)[r_3(s)[\xi(\alpha(s))]^p+r_4(s)[\eta(\alpha(s))]^p]\mathrm{d}s\big), & p=1,\end{cases}$$

$$F(t)=\exp\Big(E(t)\int_0^t (N(s)[r_1(s)\xi(s)+r_2(s)\eta(s)]+P(t)\big(k_1(t,m)\xi(m)+k_2(t,m)\eta(m)\big))\mathrm{d}s\Big)$$

$$G(t)=\begin{cases}\big[1+(1-q)\int_0^{\beta^{-1}(t)} E(t)F(t)P(t)\bar{m}(t,s)\mathrm{d}s\big]^{\frac{1}{1-q}}, & 0<q<1;\\ \exp\big(\int_0^{\beta^{-1}(t)} E(t)F(t)P(t)\bar{m}(t,s)\mathrm{d}s\big), & q=1,\end{cases}$$

$$P(t)=\int_0^t N(s)r_5(s)\mathrm{d}s,\qquad \bar{m}(t,s)=k_3(t,s)[\xi(\beta(s))]^q+k_4(t,s)[\eta(\beta(s))]^q.$$

using the conditions 3), 4) of Theorem 1 and above inequalities, and in view of the local existence of the solution to (1), we can easily see that $\beta=+\infty$, and so the boundedness of the function $Q(t)$ on $[0,\infty)$ follows from above inequalities, and then it follows that the limits of the functions $A(t)$ and $B(t)$ exist as $t\to\infty$.

The conclusions of this Theorem are independent of the choice of the solutions $Z_1(t)$ and $Z_2(t)$. In fact, if $\bar{Z}_1(t)$ and $\bar{Z}_2(t)$ are another two linearly independent solutions of (3), then there exist some constants c_i, $i=1,2,3,4$, such that $\bar{Z}_1(t)=c_1Z_1(t)+c_2Z_2(t)$, $\bar{Z}_2(t)=c_3Z_1(t)+c_4Z_2(t)$ when $t\in R_+$ and here we have

$$C=\sum_{i=1}^{4}|c_i|>0.$$

Therefore, we have $\bar{\xi}(t)=|\bar{Z}_1(t)|+|\bar{Z}_2(t)|\leqslant C\xi(t)$, $\bar{\eta}(t)=|\bar{Z}_1'(t)|+|\bar{Z}_2'(t)|\leqslant C\eta(t)$ for $t\in R_+$ and the Wronskian determinant $\bar{W}(t)$ formed by $\bar{Z}_1$, $\bar{Z}_2$ can be written as $\bar{W}(t)=KW(t)$, $t\in R_+$, where $K\neq 0$ is a constant. Hence we can easily see that the applying conditions 3), 4) of Theorem 1 also hold when $\xi(t)$, $\eta(t)$ and $W(t)$ are replaced by $\bar{\xi}(t)$, $\bar{\eta}(t)$ and $\bar{W}(t)$ respectively. The proof is completed.

Remark 2 If $k_1(t,s)\equiv k_2(t,s)\equiv 0$, Theorem 1 implies the main results in [1].

Remark 3 If $k_1(t,s)\equiv k_2(t,s)\equiv 0$ and $r_3(t)\equiv r_4(t)\equiv 0$ Theorem 1 implies the main results in [3].

Remark 4 If $k_1(t,s)\equiv k_2(t,s)\equiv 0$ and $r_5(t)\equiv 0$ or

$$f(t,\,x_1,\,x_2,\,x_3,\,x_4,\,x_5)\equiv f(t,\,x_1,\,x_2,\,x_3,\,x_4),$$

Theorem 1 implies the main results in [4].

Corollary 1 Consider the equation

$$\begin{aligned}(ax')'+bx'+cx &= \theta(t)\int_0^t u(t,s)x(s)\mathrm{d}s+\omega(t)\int_0^t v(t,s)[x'(\beta(s))]^q\mathrm{d}s\\ &\quad +h(t,x(t),x'(t),x(\alpha(t)),x'(\alpha(t)))\end{aligned}\tag{25}$$

where $p\in(0,1]$ is a constant, and $a(t)$, $b(t)$, $c(t)$, $\alpha(t)$, $\beta(t)$ are as the same functions as defined in Theorem 1, $|u(t,s)|$, $|v(t,s)|$ are nondecreasing in t for $u(t,s)$, $v(t,s)\in C(R_+^2,R)$ and for each fixed s and $h(t,\,x_1,\,x_2,\,x_3,\,x_4)\in C(R_+\times R^4,R)$, $\theta(t)$, $\omega(t)\in C(R_+,R)$. we suppose here that

1) when $t\in R$, $x_1,x_2,x_3,x_4\in R$, we have

$$|h(t,x_1,x_2,x_3,x_4)|\leqslant f_1(t)|x_1|+f_2(t)|x_2|+f_3(t)|x_3|^p+f_4(t)|x_4|^p+f_5(t),$$

here $0<p\leqslant 1$, $f_i\in C(R_+,R_+)$, $i=1,2,3,4,5$;

2) The function $N(t)|\theta(t)|$, $N(t)f_5(t)$, $N(t)f_1(t)\xi(t)$, $N(t)f_2(t)\eta(t)$, and $N(t)f_3(t)\xi^p(\alpha(t))$, $N(t)f_4(t)\eta^p(\alpha(t))$, belong to $L_1(0,\infty)$, here $\xi(t)$, $\eta(t)$, $N(t)$ are defined as above;

3) when $t\to\infty$ we have

$$\int_0^t|u(t,s)|\xi(s)\mathrm{d}s<\infty,\quad \int_0^t|v(t,s)|\eta^q(\beta(s))\mathrm{d}s<\infty.$$

Then the same conclusions as stated in Theorem 1 are valid. In fact, we easily verify by letting

$$g(t,\, s,\, x_1,\, x_2,\, x_3,\, x_4) = \theta(t)\int_0^t u(t,s)x(s)\mathrm{d}s + \omega(t)\int_0^t v(t,s)[x'(\beta(s))]^q \mathrm{d}s,$$

and

$$\begin{aligned} f(t,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5) &= \theta(t)\int_0^t u(t,s)x(s)\mathrm{d}s + \omega(t)\int_0^t v(t,s)[x'(\beta(s))]^q \mathrm{d}s \\ &\quad + h(t, x(t), x'(t), x(\alpha(t)), x'(\alpha(t))) \end{aligned}$$

in the above theorem.

Example 1 We consider the equation

$$\begin{aligned} & x'' + 3x' + 2x \\ = & \int_0^t \mathrm{e}^{-t}u(t,s)x(s)\mathrm{d}s + (1-\sin t)\mathrm{e}^{-t}x'(t) - 2\mathrm{e}^{-4t}x(t) + (1+\cos t)\mathrm{e}^{-t}[x'(t-\pi)]^{\frac{1}{2}} \\ & + \mathrm{e}^{-4t}[x(t-\pi)]^{\frac{1}{2}} + \frac{\mathrm{e}^{-t}}{1+t^3} + \int_0^t v(t,s)[x'(s-2)]^{\frac{1}{3}}\mathrm{d}s, \quad t \in R_+. \end{aligned} \tag{26}$$

According to Corollary 1, here we have

$$a(t) = 1, \quad q = \frac{1}{3}, \quad \theta(t) = \mathrm{e}^{-t}, \quad \omega(t) = 1, \quad f_1 = 2\mathrm{e}^{-4t}, \quad f_2 = 2\mathrm{e}^{-t},$$

$$f_3 = 2\mathrm{e}^{-t}, \quad f_4 = \mathrm{e}^{-4t}, \quad f_5 = \frac{\mathrm{e}^{-t}}{1+t^3}, \quad \alpha(t) = t-2, \quad \beta(t) = t-\pi, \quad p = \frac{1}{2}.$$

If we take $Z_1(t) = \mathrm{e}^{-2t},\ Z_2(t) = \mathrm{e}^{-t}$, then

$$W(t) = \mathrm{e}^{-3t}, \quad \xi(t) = \mathrm{e}^{-t} + \mathrm{e}^{-2t}, \quad \eta(t) = \mathrm{e}^{-t} + 2\mathrm{e}^{-2t}, \quad N(t) = \mathrm{e}^{2t} + \mathrm{e}^{t},$$

so that, if $u(t,s),\ v(t,s) : R_+^2 \to R$ is bounded continuous function with $|u(t,s)|,\ |v(t,s)|$ non-decreasing in t for each fixed s, then all conditions of the Corollary 1 are satisfied, and then for every interval function $\theta(t)$ defined on $t \in [-2,0]$, the solution of the equation (26) satisfying the initial condition $x(t) = \theta(t),\ t \in [-2,0]$ approaches to zero when $t \to \infty$.

4 Applications

Consider the following equation

$$\begin{aligned} x''(t) + q(t)x(t) = f\Big[t,\, x(t),\, x'(t),\, x(\alpha(t)),\, x'(\alpha(t)), \\ \int_0^t g(t,s,x(s),x'(s),x(\beta(s)),x'(\beta(s)))\mathrm{d}s\Big], \end{aligned} \tag{27}$$

where f satisfies the conditions 1)-4) of Theorem 1 and

5) $q(t)$ tends monotonically to the positive constant ω^2 as $t \to \infty$, $q'(t) \geqslant 0$ (or $q'(t) \leqslant 0$) for all $t \in R_+$.

Theorem 2 If the conditions 1)-4) of Theorem 1 and condition 5) are satisfied then every solution of (27) is either oscillatory with the sequence of amplitudes tending to a finite limit or tends to zero as $t \to \infty$.

Proof Consider the equation

$$y''(t) + q(t)y(t) = 0. \tag{28}$$

It is clear that this equation is oscillatory[1] and every solution has the following property. If $\{t_n\}$ is the sequence of the zeros of the derivative $y'(t)$, the sequence of the amplitudes $\{|y(t_n)|\}$ has a positive limit. In fact, consider the Liapunov function

$$W(t) = \frac{1}{q(t)} y'^2(t) + y^2(t) > 0$$

By differentiation and using (28), we see that

$$W'(t) = -\frac{q'(t)}{q^2(t)} y'^2(t) \leqslant 0.$$

Hence $W(t)$ is monotonic decreasing and bounded below. It follows that every solution of (28) and its derivative are bounded. Moreover, $\lim W(t) = \alpha^2$ exists as $t \to \infty$. Consequently, $y^2(t_n) \to \alpha^2$, i.e., $|y(t_n)| \to |\alpha|$ as $n \to \infty$. It remains to show that $\alpha^2 \neq 0$. For this, consider the function $V(t) = y'^2(t) + q(t)y^2(t) > 0$. i.e. $V(t)$ is monotonic increasing to some positive constant β^2. From the equality $W(t) = \frac{1}{q(t)} V(t)$, it follows that $\alpha^2 = \frac{1}{\omega^2}\beta^2 > 0$. Let $Z_1(t)$, $Z_2(t)$ be two linearly independent solutions of (28), then according to Theorem 1. Every solution of (27) can be written in the form

$$x(t) = [a_1 Z_1(t) + a_2 Z_2(t)] + o(1), \quad t \to \infty, \tag{29}$$

where a_1, a_2 are constants. Since (28) is linear, we can write (29) in the form

$$x(t) = y(t) + o(1), \quad t \to \infty, \tag{30}$$

where $y(t)$ is a solution of (28). From Theorem 2, we see that the sequence of the amplitudes of $y(t)$ tends to a positive constants α^2 (depending on y). The relation (29) gives the required result. In fact, the first possibility, i.e., the oscillation of the solution $x(t)$ occurs when $(a_1, a_2) \neq (0, 0)$ and the second, i.e., the tendency to zero occurs when $(a_1, a_2) = (0, 0)$. The second case of the Theorem when $q'(t) \leqslant 0$ can be similarly treated.

Now consider the following special case of (1)

$$\begin{aligned} x''(t) + \omega^2 x(t) &= f\Big[t, x(t), x'(t), x(\varphi(t)), x'(\varphi(t)), \\ &\qquad \int_0^t g(t, s, x(s), x'(s), x(\varphi(s)), x'(\varphi(s)))\mathrm{d}s\Big], \end{aligned} \tag{31}$$

where f satisfies the same condition and $\omega^2 > 0$ is a constant.

Theorem 3 Every solution of (31) is oscillatory and can be written in the form $x(t) = A(t)\sin(\omega t + \delta(t))$, where $\lim A(t)$, $\lim \delta(t)$ exist as $t \to \infty$, or $x(t)$ tends to zero as $t \to \infty$.

Proof Consider the equation $y''(t) + \omega^2 y(t) = 0$, which has two linearly independent solutions $Z_1(t) = \sin\omega t$, $Z_2(t) = \cos\omega t$, for which $\frac{1}{\omega}$, and $N(s)$, $\xi(s)$ and $\eta(s)$ are bounded. Consequently, Theorem 1 implies

$$x(t) = A_1(t)\sin\omega t + A_2(t)\cos\omega t, \tag{32}$$

where $\lim A_1(t)$, $\lim A_2(t)$ exist as $t \to \infty$. If one of these limits, say $\lim A_2(t)$, is not equal to zero then (32) can be written in the form $x(t) = A(t)\sin(\omega t + \delta(t))$, where

$$A(t) = [A_1^2(t) + A_2^2(t)]^{\frac{1}{2}}, \quad \delta(t) = \arctan\left[\frac{A_1(t)}{A_2(t)}\right].$$

Otherwise, $x(t)$ has the form (32) and $\lim_{t\to\infty} x(t) = 0$.

Corollary 2 From Theorem 2, it follows that for every oscillatory solution of (31) the limit of the sequence of the zero points t_n of the solution has the asymptotic behavior $t_n = \frac{\pi}{\omega}n + o(1)$. This generalizes a result obtained in [7] for linear equation $x''(t) + \omega^2 x(t) + r_1(t)x(t) + r_2(t)x(\alpha(t)) = 0$. The technique here is different and, comparatively, simple.

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二阶具有偏差变元的积分微分方程解的渐近性

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摘　要: 本文通过建立一些新的积分不等式，解决了一类二阶具有偏差变元的积分微分方程解的渐近性和振动性。

关键词: 解的渐近行为；二阶积分微分方程；积分不等式